Foundations of Software Science (ソフトウェア基礎科学) Week 6-7, 2019

Instructors: Kazutaka Matsuda and Eijiro Sumii

Notice!

In the next week, we will have a report assignment.

Curry-Howard Correspondence

Intuitionistic Logic

Let us consider the following proposition and proof.

Proposition. There is an irrational a such that a^b is rational for an irrational b.
Proof. Let us consider a number $\sqrt{2}^{\sqrt{2}}$. If $\sqrt{2}^{\sqrt{2}}$ is irrational, then take $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. If $\sqrt{2}^{\sqrt{2}}$ is rational, then take $a = \sqrt{2}$ and $b = \sqrt{2}$.

This proof is correct in the *classical* logic, but does not exactly give us those a and b. In other words, the proof is *not constructive*. In contrast, we can prove the proposition by taking $a = \sqrt{2}$ and $b = 2 \log 3$ (in this case $a^b = 3$). This proof is *constructive* in the sense that we have constructed this concrete pair of a and b.

The *intuitionistic logic*, very roughly speaking, is a (sort of) logic that allows only constructive proofs: $P \vee Q$ has a proof only when we have a proof of P or a proof of Q, and $\exists x.P(x)$ has a proof only when we find a concrete a such that $P(a)$ holds. This is quite different from the classical logic, in which we can prove $P \vee \neg P$ without proving P or $\neg P$. Notice that we used the fact that $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational in the above proof.

Minimal (Propositional) Logic

Here, the set of propositional formulas is defined as follows.

$$A ::= P \mid A_1 \Rightarrow A_2 \mid A_1 \wedge A_2 \mid A_1 \vee A_2$$

Sometimes, we consider a special propositional letter $\perp$.

We will give a set of deduction rules for the minimal logic, a negation-free fragment of intuitionistic propositional logic, in the natural deduction style. The judgment $\Delta \vdash A$ explicitly includes the assumptions as Δ:

which is read that under a set Δ of assumptions A holds.

$$\frac{A \in \Delta}{\Delta \vdash A}\,\text{Ax} \qquad \frac{\Delta \cup \{A_1\} \vdash A_2}{\Delta \vdash A_1 \Rightarrow A_2}\Rightarrow\text{-I} \qquad \frac{\Delta \vdash A_1 \Rightarrow A_2 \quad \Delta \vdash A_1}{\Delta \vdash A_2}\Rightarrow\text{-E}$$

$$\frac{\Delta \vdash A_1 \quad \Delta \vdash A_2}{\Delta \vdash A_1 \wedge A_2}\wedge\text{-I} \qquad \frac{\Delta \vdash A_1 \wedge A_2}{\Delta \vdash A_1}\wedge\text{-E}_1 \qquad \frac{\Delta \vdash A_1 \wedge A_2}{\Delta \vdash A_2}\wedge\text{-E}_2$$

$$\frac{\Delta \vdash A_1}{\Delta \vdash A_1 \vee A_2}\vee\text{-I}_1 \qquad \frac{\Delta \vdash A_2}{\Delta \vdash A_1 \vee A_2}\vee\text{-I}_2 \qquad \frac{\Delta \vdash A_1 \vee A_2 \quad \Delta \cup \{A_1\} \vdash A' \quad \Delta \cup \{A_2\} \vdash A'}{\Delta \vdash A'}\vee\text{-E}$$

We say that A is *provable* under assumptions Δ if $\Delta \vdash A$ is derivable (i.e., there is a derivation tree whose root concludes $\Delta \vdash A$), and especially when $\Delta = \emptyset$, we just say that A is *provable*. Some examples of deducible formula are $((P \Rightarrow Q) \wedge P) \Rightarrow Q$ and $(P \wedge Q) \Rightarrow (Q \wedge P)$. A derivation tree in a proof system is sometimes called a *proof tree*.

Exercise. Write proof trees for $((P \Rightarrow Q) \wedge P) \Rightarrow Q$ and $(P \wedge Q) \Rightarrow (Q \wedge P)$. □

Exercise. Write a proof tree for $((P \Rightarrow Q) \Rightarrow P) \wedge (P \Rightarrow (P \Rightarrow Q)) \Rightarrow Q$. □

Adding the following rule, we will obtain a proof system of the *intuitionistic propositional logic.*

$$\frac{\Delta \vdash \bot}{\Delta \vdash A}\bot\text{-E}$$

The rule is sometimes called *ex falso quodlibet* ("from falsehood, anything"). Then, negation $\neg A$ is given as a shorthand for $A \Rightarrow \bot$. Check that $\Delta \vdash A$ and $\Delta \vdash A \Rightarrow \bot$ implies $\Delta \vdash \bot$, and $\Delta, A \vdash \bot$ implies $\Delta \vdash A \Rightarrow \bot$.

Adding *either* one of the following rules to the proof system of the intuitionistic propositional logic, we will obtain a proof system for the *classical propositional logic.*

$$\frac{\Delta \vdash (A \Rightarrow \bot) \Rightarrow \bot}{\Delta \vdash A}\,\text{DNE} \qquad \frac{}{\Delta \vdash A \vee (A \Rightarrow \bot)}\,\text{EM} \qquad \frac{\Delta \cup \{A \Rightarrow \bot\} \vdash \bot}{\Delta \vdash A}\,\text{PBC}$$

The system is known to be sound and complete; i.e., A is provable if and only if A is a tautology. Also, it is known that A is provable in the classical logic if and only if $\neg\neg A$ is provable in the intuitionistic logic.

Exercise. Write a proof tree of $((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$ in the classical propositional logic.

Curry-Howard Correspondence

We define the function $\phi(-)$ from types to propositional formulas as follows.

$$\begin{aligned} \phi(P) &= P && (P \text{ is a base type}) \\ \phi(\tau_1 \to \tau_2) &= \phi(\tau_1) \Rightarrow \phi(\tau_2) \\ \phi(\tau_1 \times \tau_2) &= \phi(\tau_1) \wedge \phi(\tau_2) \\ \phi(\tau_1 + \tau_2) &= \phi(\tau_1) \vee \phi(\tau_2) \end{aligned}$$

Clearly, $\phi(-)$ is a bijection. Also, we define the function erase$(-)$ as follows.

$$\mathrm{erase}(\Gamma) = \{\phi(\tau) \mid (x,\tau) \in \Gamma\}$$

Then, we have a following theorem.

Theorem (Curry-Howard Correspondence)**.**

- For any Γ, M and τ, $\Gamma \vdash M : \tau$ implies $\mathrm{erase}(\Gamma) \vdash \phi(\tau)$.
- For any Δ and A, there exist Γ and M such that $\Delta \vdash A$ implies $\Gamma \vdash M : \phi^{-1}(A)$, and $\Delta = \mathrm{erase}(\Gamma)$.

We do not show the complete proof (that can be done by induction on the derivations), but show intuition underlying the proof by writing corresponding typing/deduction rules side by side, as follows.

$$\frac{(x,\tau)\in\Gamma}{\Gamma\vdash x:\tau}\;\textsc{T-Var}\qquad\frac{A\in\Delta}{\Delta\vdash A}\;\textsc{Ax}$$

$$\frac{\Gamma\uplus\{x\mapsto\tau_1\}\vdash M:\tau_2}{\Gamma\vdash\lambda x.M:\tau_1\to\tau_2}\;\textsc{T-Abs}\qquad\frac{\Delta\cup\{A_1\}\vdash A_2}{\Delta\vdash A_1\to A_2}\;\Rightarrow\text{-I}$$

$$\frac{\Gamma\vdash M:\tau_1\to\tau_2\quad\Gamma\vdash N:\tau_1}{\Gamma\vdash M\;N:\tau_2}\;\textsc{T-App}\qquad\frac{\Delta\vdash A_1\Rightarrow A_2\quad\Delta\vdash A_1}{\Delta\vdash A_2}\;\Rightarrow\text{-E}$$

$$\frac{\Gamma\vdash M:\tau_1\quad\Gamma\vdash N:\tau_2}{\Gamma\vdash(M,N):\tau_1\times\tau_2}\;\textsc{T-Pair}\qquad\frac{\Delta\vdash A_1\quad\Delta\vdash A_2}{\Delta\vdash A_1\wedge A_2}\;\wedge\text{-I}$$

$$\frac{\Gamma\vdash M:\tau_1\times\tau_2}{\Gamma\vdash\pi_1 M:\tau_1}\;\textsc{T-Fst}\qquad\frac{\Delta\vdash A_1\wedge A_2}{\Delta\vdash A_1}\;\wedge\text{-E}_1$$

$$\frac{\Gamma\vdash M:\tau_1\times\tau_2}{\Gamma\vdash\pi_2 M:\tau_2}\;\textsc{T-Snd}\qquad\frac{\Delta\vdash A_1\wedge A_2}{\Delta\vdash A_2}\;\wedge\text{-E}_2$$

$$\frac{\Gamma\vdash M:\tau_1}{\Gamma\vdash\mathsf{InL}\;M:\tau_1+\tau_2}\;\textsc{T-Left}\qquad\frac{\Delta\vdash A_1}{\Delta\vdash A_1\vee A_2}\;\vee\text{-I}_1$$

$$\frac{\Gamma\vdash M:\tau_2}{\Gamma\vdash\mathsf{InR}\;M:\tau_1+\tau_2}\;\textsc{T-Right}\qquad\frac{\Delta\vdash A_2}{\Delta\vdash A_1\vee A_2}\;\vee\text{-I}_2$$

$$\frac{\begin{array}{c}\Gamma\vdash M:\tau_1+\tau_2\\ \Gamma\uplus\{x\mapsto\tau_1\}\vdash N_1:\tau'\\ \Gamma\uplus\{y\mapsto\tau_2\}\vdash N_2:\tau'\end{array}}{\Gamma\vdash\mathbf{case}\;M\;\mathbf{of}\;(x.N_1)\;(y.N_2):\tau'}\;\textsc{T-Case}\qquad\frac{\begin{array}{c}\Delta\vdash A_1\vee A_2\\ \Delta\cup\{A_1\}\vdash A'\\ \Delta\cup\{A_2\}\vdash A'\end{array}}{\Delta\vdash A'}\;\vee\text{-E}$$

In this sense, a λ-term represents a proof in the minimal logic. Such a term representing a proof is sometimes called a *proof term.*

Exercise. Give λ-terms of the types $((P \to Q) \times P) \to Q$ and $(P \times Q) \to (Q \times P)$, where P and Q are some base types. Show their typing derivations. □

Exercise. Give a λ-term of the type $((P \to Q) \to P) \wedge (P \to (P \to Q)) \to Q$, where P and Q are some base types. Show its typing derivation. □

Exercise. Give a λ-term of the type $((P + (P \to R)) \to R) \to R$, where P and R are some types. Show its typing derivation. □

Consistency via Curry-Howard

Definition (Consistency)**.** We call a proof system *consistent* if $\bot$ is not provable in the system. □

Theorem. The minimal logic is consistent.

Proof. Suppose that $\emptyset \vdash \bot$. Then, by Curry-Howard correspondence, we have a term M such that $\emptyset \vdash M : \bot$. (That is, the simply-typed λ-calculus is a model of the minimal logic.) By the subject reduction property and strong normalization, we have a value V such that $\emptyset \vdash V : \bot$. Then, we perform case analysis of the form of V to show that V cannot have type $\bot$ for each case.

If V has the form of $\lambda x.V'$ for some V'. Its type must has the form of $\tau_1 \to \tau_2$, which cannot be $\bot$. Similar discussions apply to the cases $V = (V_1, V_2)$, $V = \mathsf{InL}\ V'$, and $V = \mathsf{InR}\ V'$. Then, we consider the case where $V = W$ for a neutral term W. However, it is easy to show that $\mathrm{FV}(W) \neq \emptyset$ and W cannot have any type under the empty type environment. □

In the proof, subjection reduction and strong normalization play very important roles in showing consistency. Generally speaking, these two properties are important in a proof calculus based on Curry-Howard correspondence, such as the calculus of construction and the Martin-Löf type theory. Both are (quite big) extension of the simply-typed λ-calculus, and underlie proof assistants Coq and Agda, respectively.

FYI: Simply-Typed λ-Calculus for Intuitionistic Propositional Logic with Negation

We add following construct to the simply-typed λ-calculus with pairs and sums.

$$M, N ::= \cdots \mid \mathsf{error}\ M$$

Accordingly, we add $\bot$ to the set of types. Its reduction rules are given by:

$$\frac{}{(\mathsf{error}\ M)\ N \longrightarrow \mathsf{error}\ M} \quad \frac{}{\pi_i\ (\mathsf{error}\ M) \longrightarrow \mathsf{error}\ M}$$

$$\frac{}{\textbf{case}\ (\mathsf{error}\ M)\ \textbf{of}\ (x.N_1)\ (y.N_2) \longrightarrow \mathsf{error}\ M} \quad \frac{}{\mathsf{error}\ (\mathsf{error}\ M) \longrightarrow \mathsf{error}\ M}$$

alongside with

$$\frac{M \longrightarrow M'}{\mathsf{error}\ M \longrightarrow \mathsf{error}\ M'}$$

Intuitively, error propagates the "exception" that something wrong has happened.

We also extend the set of values.

$$V ::= \cdots \mid \mathsf{error}\ W$$

The typing rule for error is designed as a correspondent to the rule ⊥-E.

$$\frac{\Gamma \vdash M : \bot}{\Gamma \vdash \mathsf{error}\ M : \tau}\ \text{T-Error}$$

Still we have many important properties, including subject reduction, progress, and strong normalization. Also, check that there is no value that has type $\bot$ under the empty type environment.

Exercise. Give a λ-term of type $((P \to Q) \to P) \to ((P \to \bot) \to \bot)$. Show its typing derivation.

FYI: Kripke Model of Intuitionistic Logic

In the classical logic, a proposition is either true or false. However, in the intuitionistic logic, things are not that simple; there is a proposition A such that A is not provable whereas $\neg A$ leads to contradiction (i.e., $\neg\neg A$ is provable). So, what does a proposition in the intuitionistic logic *represent*?

Curry-Howard correspondence suggests that a proposition A represents a set of proofs for A: a proof for $A_1 \wedge A_2$ is a pair of proofs for A_1 and A_2 respectively, a proof for $A_1 \vee A_2$ is either a left-tagged proof of A_1 or a right-tagged proof of A_2, and a proof for $A_1 \Rightarrow A_2$ is a function that maps a proof of A_1 to a proof of A_2. This view of propositions is known as the *BHK* interpretation, which clarify the constructive nature of the intuitionistic logic.

Here, we introduce another model of the intuitionistic propositional logic called the Kripke model.

Definition (Kripke model)**.** A Kripke model is a triple $(W, \preceq, \Vdash)$ of a non-empty set W, a partial-order $\preceq$ on W, and binary relation $\Vdash$ between W and propositional formulas, satisfying the following conditions for any $w \in W$ and any propositions A_1 and A_2.

- if $w \preceq w'$, $w \Vdash P$ implies $w' \Vdash P$ for any propositional letter P.
- $w \Vdash A_1 \wedge A_2$ if and only if $w \Vdash A_1$ and $w \Vdash A_2$.
- $w \Vdash A_1 \vee A_2$ if and only if either $w \Vdash A_1$ or $w \Vdash A_2$.
- $w \Vdash A_1 \Rightarrow A_2$ if and only if $w' \Vdash A_2$ for any w' such that $w \preceq w'$ and $w' \Vdash A_1$.
- $w \Vdash \bot$ does not hold for any $w \in W$. □

Elements of W are sometimes called *world*. Intuitively, $w \preceq w'$ means that w' is a future of w, and $w \Vdash A$ means that we *know* that A holds at world w. The first line says that, once we know that P holds, we also know that P holds in any future. The condition of $w \Vdash A_1 \Rightarrow A_2$ means that if we know that A_1 holds now or in some future, A_2 must hold from this point. Notice that, for any world w, we have that $w \Vdash \neg A$ if and only if $w' \nVdash A$

for any world w' such that $w \preceq w'$. In other words, A does not hold neither now nor in the future. We omit the last line if we consider the minimal logic instead of the full intuitionistic propositional logic.

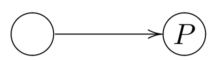
Sometimes, for a Kripke model $\mathcal{M} = (W, \preceq, \Vdash)$, we write $\mathcal{M}, w \Vdash A$ to clarify the model we consider. For a Kripke model $\mathcal{M} = (W, \preceq, \Vdash)$, we write $\mathcal{M}, w \Vdash \Delta$ if $\mathcal{M}, w \Vdash A$ for all $A \in \Delta$. Also, we write $\Delta \Vdash A$ if for every Kripke model $\mathcal{M} = (W, \preceq, \Vdash)$ and world $w \in W$, $\mathcal{M}, w \Vdash \Delta$ implies $\mathcal{M}, w \Vdash A$.

It is known that the proof system of the intuitionistic propositional logic is sound and complete with respect to the Kripke model.

Theorem. $\Delta \vdash A$ if and only $\Delta \Vdash A$. □

Model theory is sometimes useful to show that a certain proposition is not provable.

Example(s). We show that $\Vdash ((P \Rightarrow \bot) \Rightarrow \bot)) \Rightarrow P$ by giving a concrete model $\mathcal{M} = (W, \preceq, \Vdash)$ such that $w \nVdash ((P \Rightarrow \bot) \Rightarrow \bot)) \Rightarrow P$ for some $w \in W$. To give such a model, there must be a world such that $w \Vdash ((P \Rightarrow \bot) \Rightarrow \bot)$ but not $w \Vdash P$. The former condition means that for any world w' with $w \preceq w'$, there is some w'' such that $w' \preceq w''$ and $w'' \Vdash P$. That is, P *necessarily* holds, but P is not required to be true now. Thus, $\mathcal{M} = (\{w_1, w_2\}, \{(w_1, w_2)\}^*, \{(w_2, P)\})$ is such a model because $w_1 \Vdash (P \Rightarrow \bot) \Rightarrow \bot$ holds but $w_1 \Vdash P$ does not. We can illustrate this model as follows.



Here, a world is represented by a circle labeled by the propositional letters that hold in the world. □