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Propositional Logic

Proposition: a statement that is true or false.

Example(s). "1 + 1 + 1 is 3", "I am Matsuda", "2 is greater than 3". □

Propositional logic: a logic whose atomic constructs are proposition.

Table 1: Cheat Sheet of Propositional Logic

Formula	How to Read	Informal Explanation: When it is True
P	"P"	A variable that ranges over propositions.
$\neg A$	"not A"	A is false.
$A \wedge B$	"A and B"	Both A and B are true.
$A \vee B$	"A or B"	At least one of A and B is true.
$A \Rightarrow B$	"A implies B"	B is true whenever A is.

Name of Symbols P (propositional variable/propositional letter), $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), and $\Rightarrow$ (implication).

Definition. A formula A is a *tautology* if A is true no matter of the truth of propositional variables in it. □

Example(s). $P \Rightarrow P$, $P \wedge Q \Rightarrow P$, $P \vee \neg P$, $\neg(P \wedge Q) \Rightarrow \neg P \vee \neg Q$ are tautologies. □

Predicate Logic

Predicate: a statement with (zero or more) variables for things (individials) that becomes true or false after substituting the variables with concrete individuals.

Example(s). "x is 3", "x is Matsuda", "x is greater than y". □

Predicate logic: a logic whose atomic constructs are predicates.

Table 2: Cheat Sheet of Predicate Logic

Formula	How to Read	Informal Explanation: When it is True
$P(x_1, \ldots, x_n)$	"$P(x_1, \ldots, x_n)$"	A variable that represents a predicate with variables $x_1, \ldots, x_n$.
$\forall x.A$	"For any x, A"	A is true for all individuals x.
$\exists x.A$	"There exists x s.t. A"	B is true for some individual x.

We also use individual constant a, b, c, etc. For some specific theories, we may write $\forall x \in X.A$ or $\exists x \in X.A$ to specify the set that x ranges over.

Note. Nullary predicates (or, predicates with zero variables) are propositions.

Name of Symbols $\forall$ (universal quantifier), and $\exists$ (existential quantifier).

Definition. A formula A is *valid* if A is true no matter how we replace the individual constants in A with concrete individuals and the predicate variables in A with concrete predicates.

Note. The set of individuals must be instantiated to a non-empty set. This the reason why $(\forall x.P(x)) \Rightarrow (\exists x.P(x))$ is valid.

Example(s). $P(a) \Rightarrow \exists x.P(x)$, and $(\exists x.\forall y.P(x,y)) \Rightarrow (\forall y.\exists x.P(x,y))$ are valid. Note that the converse of the latter predicate, $(\forall y.\exists x.P(x,y)) \Rightarrow (\exists x.\forall y.P(x,y))$, is not valid. □

Some Notations for Set

Notation	Meaning
$S \cap T$	$\forall x.\, x \in (S \cap T) \Leftrightarrow x \in S \wedge y \in T$.
$S \cup T$	$\forall x.\, x \in (S \cup T) \Leftrightarrow x \in S \vee y \in T$.
$S \setminus T$	$\forall x.\, x \in (S \setminus T) \Leftrightarrow x \in S \wedge \neg(x \in T)$.
$S \subseteq T$	$S \subseteq T \Leftrightarrow \forall x.\, x \in S \Rightarrow x \in T$.
$S = T$	$S = T \Leftrightarrow S \subseteq T \wedge T \subseteq S$.
2^S, $\mathcal{P}(S)$	$\forall x.\, x \in 2^S \Leftrightarrow x \subseteq S$.
$\{x \in X \mid P(x)\}$	$\forall y.\, y \in \{x \in X \mid P(x)\} \Leftrightarrow y \in X \wedge P(y)$.

Sometimes, we write $\{x \mid x \in X \wedge P(x)\}$ or $\{x \mid x \in X, P(x)\}$ for $\{x \in X \mid P(x)\}$.

Mathematical Induction

We write $\mathbb{N}$ for the set of natural numbers. (In logic and computer science, 0 is a natural number.)

Axiom (Induction Principle on Natural Numbers)**.** For all unary predicates P on $\mathbb{N}$,

$$\forall x \in \mathbb{N}.\, P(x) \quad \Leftrightarrow \quad \big(P(0) \wedge \forall x \in \mathbb{N}.P(x) \Rightarrow P(x+1)\big)$$

holds. □

Theorem (Complete Induction)**.** For all unary predicates P on $\mathbb{N}$,

$$\forall x \in \mathbb{N}.\, P(x) \quad \Leftrightarrow \quad \big(\forall x \in \mathbb{N}.(\forall y \in \mathbb{N}.(y < x) \Rightarrow P(y)) \Rightarrow P(x)\big)$$

holds.

Proof. Apply the induction principle for the predicate $Q(x) = \forall y \in \mathbb{N}.(y \leq x) \Rightarrow P(y)$. □

Definition (Well-founded Relation)**.** A relation $\prec$ on S is *well-founded* if there is no infinite sequence $x_1, x_2, \ldots$ in S such that $x_{i+1} \prec x_i$ for all $i \geq 1$. □

Theorem. For all well-founded relations $\prec$ on S and all unary predicates P,

$$\forall x \in S.P(x) \quad \Leftrightarrow \quad \big(\forall x \in S.(\forall y \in S.(y \prec x) \Rightarrow P(y)) \Rightarrow P(x)\big)$$

holds.